Given the three sides of a triangle are *a*, *ar* and *ar*^{2}, find the range of *r* .

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The lengths of sides of a triangle must satisfy triangle inequality: the sum of two is greater than the third.

$$\begin{cases} a + ar > ar^2 \\ a + ar^2 > ar \\ ar + ar^2 > a \end{cases} \Rightarrow \begin{cases} 0 > r^2 - r - 1 & \cdots\cdots(1) \\ r^2 - r + 1 > 0 & \cdots\cdots(2) \\ r^2 + r - 1 > 0 & \cdots\cdots(3) \end{cases}$$

From (1): $0 > \left(r - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2$

$0 > \left(r - \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(r - \frac{1}{2} - \frac{\sqrt{5}}{2}\right)$

$\frac{1}{2} - \frac{\sqrt{5}}{2} < r < \frac{1}{2} + \frac{\sqrt{5}}{2}$ $\cdots\cdots$ (4)

From (2): $\left(r - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 > 0$

Always true, r can be any real numbers $\cdots\cdots$ (5)

From (3): $\left(r + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2 > 0$

$\left(r + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(r + \frac{1}{2} - \frac{\sqrt{5}}{2}\right) > 0$

$r < -\frac{1}{2} - \frac{\sqrt{5}}{2}$ or $-\frac{1}{2} + \frac{\sqrt{5}}{2} < r$ $\cdots\cdots$ (6)

$\because$ $-\frac{1}{2} - \frac{\sqrt{5}}{2} < \frac{1}{2} - \frac{\sqrt{5}}{2} < 0 < -\frac{1}{2} + \frac{\sqrt{5}}{2} < \frac{1}{2} + \frac{\sqrt{5}}{2}$

Combine (4), (5) and (6): $-\frac{1}{2} + \frac{\sqrt{5}}{2} < r < \frac{1}{2} + \frac{\sqrt{5}}{2}$

$0.618 < r < 1.618$ (correct to 3 significant figures)